

Weak solutions of the Shigesaka-Kawasaki-Teramoto equations and their attractors

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Abstract

WE derive the global existence of weak solutions of the Shigesada-Kawasaki-Teramoto systems in space dimension $d \leq 4$ with a rather general condition on the coefficients. The existence is established using finite differences in time with truncations and an argument of Stampachia's maximum principle to show the positivity of the solutions. We derive also the existence of a weak global attractor.

Keywords and phrases: strongly coupled reaction diffusion system; quasi-linear parabolic equations; weak global attractor; maximum principle.

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1 Introduction

WE let $\Omega \subset \mathbb{R}^d$, $d \leq 4$, be an open bounded set and denote by $\Omega_T = \Omega \times (0, T)$. We look for solutions to the following Shigesada-Kawasaki-Teramoto (SKT) system of diffusion reaction equations, see [Shigesada et al., 1979] for the setup of the system:

$$\begin{cases} \partial_t u - \Delta p_1(u, v) + q_1(u, v) = \ell_1(u) \text{ in } \Omega_T, \\ \partial_t v - \Delta p_2(u, v) + q_2(u, v) = \ell_2(u) \text{ in } \Omega_T, \end{cases} \quad (1.1)$$

$$\begin{cases} \partial_n u = \partial_n v = 0 \text{ on } \partial\Omega \times (0, T) \text{ or } u = v = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0 \text{ in } \Omega, \end{cases} \quad (1.2)$$

where

$$p_1(u, v) = (d_1 + a_{11}u + a_{12}v)u, \quad (1.3a)$$

$$p_2(u, v) = (d_2 + a_{21}u + a_{22}v)v, \quad (1.3b)$$

$$q_1(u, v) = (b_1u + c_1v)u, \ell_1(u) = a_1u, \quad (1.3c)$$

$$q_2(u, v) = (b_2u + c_2v)v, \ell_2(v) = a_2v, \quad (1.3d)$$

with $a_{ij} \geq 0, b_i \geq 0, c_i \geq 0, a_i \geq 0, d_i \geq 0$.

We will consider also the alternate form of (1.1)

$$\begin{cases} \partial_t u - \nabla \cdot [(d_1 + 2a_{11}u + a_{12}v)\nabla u + a_{12}u\nabla v] + q_1(u, v) = \ell_1(u) \text{ in } \Omega_T, \\ \partial_t v - \nabla \cdot [a_{21}v\nabla u + (d_2 + 2a_{21}u + a_{22}v)\nabla v] + q_2(u, v) = \ell_2(v) \text{ in } \Omega_T. \end{cases} \quad (1.4)$$

The mathematical literature of the SKT system (1.1) is vast; here are some highlights. In the case of weak cross-diffusion or triangular systems (when $a_{12} = 0$ or $a_{21} = 0$), there have been a series of developments: when $d = 2$, Lou et al. [Lou et al., 1998] proved the existence of smooth solutions to the SKT model. The method in [Lou et al., 1998] can also be modified to cover the case $d = 1$. Choi et al. [Choi et al., 2004] and Le et al. [Le et al., 2003] independently settled the case $d \leq 5$. Tuoc [Tuoc, 2007] proved the existence of smooth solutions to the SKT model when $d \leq 9$. Recently, Hoang et al. [Hoang et al., 2015] established the global existence of smooth solutions for any space dimension, which improves results in [Nguyen, 2006].

In the case of full systems (when $a_{12}, a_{21} \neq 0$), for solutions achieved by a maximum principle, interpolation and Sobolev embeddings, Yagi in [Yagi, 1993] proved the global existence of solutions in the two dimensional case under the condition $8a_{11} > a_{21}$, $8a_{22} > a_{12}$ by using a maximum principle developed for strong solutions $u, v \in \mathcal{C}([0, T]; H^{1+\epsilon}(\Omega)) \cap \mathcal{C}((0, T]; H^2(\Omega)) \cap \mathcal{C}^1((0, T]; L^2(\Omega))$.

In [Le, 2013, Le, 2016], the author uses a different approach by controlling the BMO norms instead of using the classical results of Amann [Amann, 1990, Amann, 1989] that require estimates for the Hölder norms of the solutions.

In this article we aim to establish the existence of weak solutions to the SKT system (1.1) in space dimension $d \leq 4$, when the coefficients satisfy

$$4a_{11} > a_{12}, \quad 4a_{22} > a_{21}, \quad (1.5a)$$

$$2a_{21} > a_{12} > \frac{1}{2}a_{21}. \quad (1.5b)$$

Our work relates to that of Yagi, [Yagi, 1993], however we deal with dimensions $d \leq 4$ while [Yagi, 1993] is limited to $\dim d = 1, 2$, and our hypotheses (1.5) are more general, perhaps the most general possible in this context.

Another difference is that our proof is completely self contained, while [Yagi, 1993] relies on some earlier articles [Yagi, 1988, Yagi, 1990, Yagi, 1991]. Our proof relies on seemingly new a priori estimates. Finally another novelty is to contract a weak global attractor for these equations in the spirit of Ball [Ball, 1997], Sell [Sell, 1996], [Foias and Temam, 1987], [Foias et al., 2001], and [Foias et al., 2010]. As in these references the concept of attractor is weak, because we do not establish the uniqueness of weak solutions, like for the 3D Navier-Stokes equations. Additional regularity of solutions and possibly the issue of uniqueness will be studied in a subsequent work.

The authors of this article are not specialists of reaction-diffusion equations and they will welcome any bibliographical reference from the Editors or Referees that they may have overlooked.

We will work with the following vector form of (1.4):

$$\partial_t \mathbf{u} - \nabla \cdot (\mathbf{P}(\mathbf{u}) \nabla \mathbf{u}) + \mathbf{q}(\mathbf{u}) = \ell(\mathbf{u}), \quad (1.6)$$

where $\mathbf{u} = (u, v)$, $\mathbf{q}(\mathbf{u}) = (q_1(\mathbf{u}), q_2(\mathbf{u}))$, $\ell(\mathbf{u}) = (\ell_1(\mathbf{u}), \ell_2(\mathbf{u}))$ and

$$\mathbf{P}(\mathbf{u}) = \begin{pmatrix} p_{11}(u, v) & p_{12}(u, v) \\ p_{21}(u, v) & p_{22}(u, v) \end{pmatrix} = \begin{pmatrix} d_1 + 2a_{11}u + a_{12}v & a_{12}u \\ a_{21}v & d_2 + a_{21}u + 2a_{22}v \end{pmatrix}. \quad (1.7)$$

We consider later on the mapping

$$\mathcal{P} : \mathbf{u} = (u, v) \mapsto \mathbf{p} = (p_1, p_2), \quad (1.8)$$

and we observe that

$$\mathbf{P}(\mathbf{u}) = \frac{D\mathcal{P}}{D\mathbf{u}}(\mathbf{u}). \quad (1.9)$$

2 Formal a priori estimates

We derive formal a priori estimates for the solutions of (1.1)-(1.6), assuming that $u, v \geq 0$ are sufficiently smooth.

2.1 First a priori estimates and conditions on the a_{ij}

By multiplying (1.6) by \mathbf{u} , integrating over Ω , and integrating by parts using the boundary condition (1.2)₁, we find

$$\frac{1}{2} \frac{d}{dt} |\mathbf{u}|^2 + \langle \mathbf{P}(\mathbf{u}) \nabla \mathbf{u}, \nabla \mathbf{u} \rangle + \langle \mathbf{q}(\mathbf{u}), \mathbf{u} \rangle = \langle \ell(\mathbf{u}), \mathbf{u} \rangle. \quad (2.1)$$

We concentrate first on the term $\langle \mathbf{P}(\mathbf{u}) \nabla \mathbf{u}, \nabla \mathbf{u} \rangle$ and for $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$ we write

$$\begin{aligned} (\mathbf{P}(\mathbf{u}) \boldsymbol{\xi}) \cdot \boldsymbol{\xi} &= (d_1 + 2a_{11}u + a_{12}v) \xi_1^2 + a_{12}u \xi_1 \xi_2 + a_{21}v \xi_1 \xi_2 + (d_2 + a_{21}u + a_{22}v) \xi_2^2 \\ &= d_1 \xi_1^2 + d_2 \xi_2^2 + (2a_{11}u + a_{12}v) \xi_1^2 + (a_{12}u + a_{21}v) \xi_1 \xi_2 + (a_{21}u + 2a_{22}v) \xi_2^2. \end{aligned} \quad (2.2)$$

When the conditions (1.5a), (1.5b) are satisfied, we can prove that the matrix $\mathbf{P}(\mathbf{u})$ is (pointwise) positive definite and that:

$$(\mathbf{P}(\mathbf{u}) \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq \alpha(u + v) |\boldsymbol{\xi}|^2 + d_0 |\boldsymbol{\xi}|^2, \quad (2.3)$$

where $d_0 = \min(d_1, d_2)$ and

$$\alpha = \min \left\{ 2a_{11} - \frac{1}{2}a_{12}, 2a_{22} - \frac{1}{2}a_{21}, a_{12} - \frac{1}{2}a_{21}, a_{21} - \frac{1}{2}a_{12} \right\} > 0. \quad (2.4)$$

Indeed, we bound from below the term $\xi_1 \xi_2$ in (2.2) as follows:

$$a_{12}u \xi_1 \cdot \xi_2 + a_{21}v \xi_1 \cdot \xi_2 \geq -\frac{1}{2}a_{12}u |\xi_1|^2 - \frac{1}{2}a_{12}u |\xi_2|^2 - \frac{1}{2}a_{21}v |\xi_1|^2 - \frac{1}{2}a_{21}v |\xi_2|^2.$$

Hence, the sum of the diffusion terms is bounded as follows

$$\begin{aligned} (\mathbf{P}(\mathbf{u}) \boldsymbol{\xi}) \cdot \boldsymbol{\xi} &\geq d_1 |\xi_1|^2 + d_2 |\xi_2|^2 + (2a_{11} - \frac{1}{2}a_{12})u |\xi_1|^2 + (a_{12} - \frac{1}{2}a_{21})v |\xi_1|^2 \\ &\quad + (2a_{22} - \frac{1}{2}a_{21})v |\xi_2|^2 + (a_{21} - \frac{1}{2}a_{12})u |\xi_2|^2 \\ &\geq d_1 |\xi_1|^2 + d_2 |\xi_2|^2 + \alpha(u + v)(|\xi_1|^2 + |\xi_2|^2). \end{aligned}$$

Remark 2.1. When $a_{12} = a_{21}$, Yagi [Yagi, 1993] shows that $\mathbf{P}(\mathbf{u})$, $u, v > 0$, is positive definite when $8a_{11} > a_{21}$ and $8a_{22} > a_{12}$. Thus (1.5) not only extends Yagi's result in the case $a_{12} = a_{21}$, but also extends to other cases when $a_{12} \neq a_{21}$ (in (1.5b)).

We thus assume the coefficients satisfy (1.5), which implies that

$$\langle \mathbf{P}(\mathbf{u}) \nabla \mathbf{u}, \nabla \mathbf{u} \rangle \geq \int_{\Omega} [d_0 + \alpha(u + v)] |\nabla \mathbf{u}|^2 dx, \quad (2.5)$$

for all $u, v > 0$.

Note that (2.3) implies that, for $u, v \geq 0$, $\mathbf{P}(\mathbf{u})$ is invertible (as a 2×2 matrix), and that, pointwise (i.e. for a.e. $x \in \Omega$),

$$|\mathbf{P}(\mathbf{u})^{-1}|_{\mathcal{L}(\mathbb{R}^2)} \leq \frac{1}{d_0 + \alpha(u + v)}. \quad (2.6)$$

We now complete the first a priori estimate (2.1). We observe that

$$\begin{aligned} \langle \mathbf{q}(\mathbf{u}), \mathbf{u} \rangle &= \int_{\Omega} [(b_1 u + c_1 v)u^2 + (b_2 u + c_2 v)v^2] dx \geq 0, \\ \langle \ell(\mathbf{u}), \mathbf{u} \rangle &= \int_{\Omega} (a_1 u^2 + a_2 v^2) dx \leq \int_{\Omega} \left(\frac{b_1}{2} u^3 + \frac{c_2}{2} v^3 \right) dx + \mathcal{K}_1, \end{aligned}$$

where \mathcal{K}_1 is an absolute constant. Finally (2.1) yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|u|^2 + |v|^2) + \int_{\Omega} [d_0 + \alpha(u + v)] (|\nabla u|^2 + |\nabla v|^2) dx \\ + \frac{1}{2} \int_{\Omega} [(b_1 u + c_1 v)u^2 + (b_2 u + c_2 v)v^2] dx \leq \mathcal{K}_1. \end{aligned} \quad (2.7)$$

Finally, (2.7) gives a priori estimates for u and v in

$$L^\infty(0, T; L^2(\Omega)) \text{ and } L^2(0, T; H^1(\Omega)), \quad (2.8)$$

and bounds of

$$\sqrt{u} \nabla u, \sqrt{u} \nabla v, \sqrt{v} \nabla u, \sqrt{v} \nabla v \text{ in } L^2(0, T; L^2(\Omega)^2). \quad (2.9)$$

We infer from (2.9) that $\sqrt{u} \nabla u, \sqrt{v} \nabla v \in L^2(L^2)$ and hence $\nabla u^{\frac{3}{2}} \in L^2(L^2)$. Now we observe that, for any $r \geq 1$

$$|\nabla \phi|_{L^2} + |\phi|_{L^r}. \quad (2.10)$$

is a norm on H^1 , which is equivalent to the usual H^1 norm (thanks to the generalized Poincaré inequality, see [Temam, 1997, Chapter 2]). Since $\left| u^{\frac{3}{2}} \right|_{L^{\frac{4}{3}}} = \left(\int_{\Omega} u^2 \right)^{\frac{3}{4}} = |u|_{L^2}^{\frac{3}{2}}$, we see that

$$u^{\frac{3}{2}} \in L^2(H^1). \quad (2.11)$$

Then depending on dimension d ,

$$u^{\frac{3}{2}} \in L^2(L^\infty) \quad d = 1, \quad (2.12a)$$

$$u^{\frac{3}{2}} \in L^2(L^q) \quad \forall q, d = 2, \quad (2.12b)$$

$$u^{\frac{3}{2}} \in L^2(L^6) \quad d = 3, \quad (2.12c)$$

$$u^{\frac{3}{2}} \in L^2(L^4) \quad d = 4. \quad (2.12d)$$

We also observe in space dimension 4 (the most restrictive case), that (2.12) implies that

$$u \in L^{\frac{3}{2}}(L^6) \quad (2.13)$$

and since $u \in L^\infty(L^2)$ we can write

$$\int_{\Omega} u^4 = \int_{\Omega} u^3 u \leq \left(\int_{\Omega} u^6 \right)^{\frac{1}{2}} \left(\int_{\Omega} u^2 \right)^{\frac{1}{2}} = |u|_{L^6}^{\frac{3}{2}} |u|_{L^2} \in L_t^1,$$

so that

$$u \in L_t^4(L^4). \quad (2.14)$$

We summarize the estimates in the following lemma

Lemma 2.1. *We assume that $\mathbf{u}_0 = (u_0, v_0) \in L^2(\Omega)^2$, $\mathbf{u}_0 \geq \mathbf{0}$, that the coefficients a_{ij} satisfy (1.5), $d \leq 4$ and that $\mathbf{u} = (u, v)$ is a smooth ≥ 0 solution of (1.1), (1.2). Then u, v satisfy (2.8), (2.9), (2.11) – (2.14). Furthermore the norms in these spaces can be bounded by constants depending on the L^2 norms of u_0, v_0 , on T and on the coefficients.*

- Remark 2.2.** 1) We refrain from giving the explicit values of the constants bounding the norms mentioned in Lemma 2.1, as the corresponding explicit calculations will not bring any further information.
- 2) We will see in Section 4 how one can improve Lemma 2.1 and derive time uniform estimates for $t \geq 0$.
- 3) The estimates in Lemma 2.1 are those that we will use for the existence of weak solutions. In Section 2.2, assuming more regularity on the initial data, we derive more a priori estimates leading to more regular solutions.

2.2 More a priori estimates

In this section, besides the assumption that $\mathbf{u}_0 = (u_0, v_0) \in L^2(\Omega)^2$ as in the previous section, we assume further that

$$\nabla \mathbf{p}(\mathbf{u}_0) \in L^2(\Omega)^4. \quad (2.15)$$

We multiply (1.1)₁ by $\partial_t p_1$ and (1.1)₂ by $\partial_t p_2$ integrate and add while noticing that

$$\begin{pmatrix} \partial_t p_1 \\ \partial_t p_2 \end{pmatrix} = \mathbf{P}(\mathbf{u}) \begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix},$$

so that we have

$$\begin{aligned} \langle \mathbf{P}(\mathbf{u}) \partial_t \mathbf{u}, \partial_t \mathbf{u} \rangle + \partial_t \left(|\nabla p_1|_{L^2}^2 + |\nabla p_2|_{L^2}^2 \right) \\ = \int_{\Omega} (q_1 \partial_t p_1 + q_2 \partial_t p_2) dx + \int_{\Omega} (\ell_1 \partial_t p_1 + \ell_2 \partial_t p_2) dx. \end{aligned} \quad (2.16)$$

By (2.3), we have

$$\langle \mathbf{P}(\mathbf{u}) \partial_t \mathbf{u}, \partial_t \mathbf{u} \rangle \geq \int_{\Omega} [d_0 + \alpha(u + v)] |\partial_t \mathbf{u}|^2 dx.$$

For the RHS of (2.16), we observe that

$$\partial_t p_1 = (d_1 + 2a_{11}u + a_{12}v) \partial_t u + a_{12}u \partial_t v$$

and the same for $\partial_t p_2$. Hence the RHS of (2.16) is bounded by

$$\int_{\Omega} (u^3 + v^3) (|\partial_t u| + |\partial_t v|) dx \leq \frac{\alpha}{2} \int_{\Omega} (u + v) (|\partial_t u|^2 + |\partial_t v|^2) dx + C \int_{\Omega} (u^5 + v^5) dx.$$

In the end

$$\int_{\Omega} [d_0 + \alpha(u + v)] |\partial_t \mathbf{u}|^2 dx + \partial_t \left(|\nabla p_1|_{L^2}^2 + |\nabla p_2|_{L^2}^2 \right) \leq C \int_{\Omega} (u^5 + v^5) dx. \quad (2.17)$$

We infer from (2.8) that

$$p_1 = d_1 u + a_{11} u^2 + a_{12} uv \in L^\infty(L^1).$$

We know from the Sobolev inclusion that

$$H^1(\Omega) \subset L^{\frac{2d}{d-2}}(\Omega)$$

with continuous embedding.

We know from (2.10) that $|\phi|_{L^1} + |\nabla \phi|_{L^2}$ is a norm on $H^1(\Omega)$ equivalent to the usual norm. Furthermore, because $u, v \geq 0$, for $d = 4$, we have

$$|u^2|_{L^4} \leq a_{11} |p_1|_{L^4} \leq C \left(|\nabla p_1|_{L^2} + |p_1|_{L^1} \right). \quad (2.18)$$

Using Hölder's inequality, we find

$$\begin{aligned} \int_{\Omega} u^5 dx &= \int_{\Omega} u \cdot u^4 dx \leq \underbrace{\left(\int_{\Omega} u^2 dx \right)^{\frac{1}{2}}}_{L_t^\infty} \left(\int_{\Omega} (u^4)^2 \right)^{\frac{1}{2}} \leq C \left(\int_{\Omega} (u^2)^4 \right)^{\frac{1}{4} \times 2} \\ &= C |u^2|_{L^4}^2 \leq (\text{using (2.18)}) \leq C \left(|\nabla p_1|_{L^2}^2 + |p_1|_{L^1}^2 \right). \end{aligned}$$

From (2.8), we see that $|p_1|_{L^1}^2 = \left(\int_{\Omega} (d_1 + a_{11}u + a_{12}v)u \right)^2 \leq C$. Thus

$$\int_{\Omega} u^5 dx \leq C \left(|\nabla p_1|_{L^2}^2 + 1 \right), \quad (2.19a)$$

and similarly

$$\int_{\Omega} v^5 dx \leq C \left(|\nabla p_2|_{L^2}^2 + 1 \right). \quad (2.19b)$$

Combing (2.17), (2.19a) and (2.19b), we have for the least easy case $d = 4$:

$$\begin{aligned} \int_{\Omega} \left[d_0 + \alpha(u+v) \right] |\partial_t \mathbf{u}|^2 dx + \partial_t \left(|\nabla p_1|_{L^2}^2 + |\nabla p_2|_{L^2}^2 \right) \\ \leq C \left(|\nabla p_1|_{L^2}^2 + |\nabla p_2|_{L^2}^2 + 1 \right). \end{aligned} \quad (2.20)$$

Using Gronwall's inequality for the function $\mathcal{Y}(t) = |\nabla p_1(t)|_{L^2}^2 + |\nabla p_2(t)|_{L^2}^2$, we conclude that $\mathcal{Y}(t)$ is bounded on $[0, T]$ by a constant, independent of t and dependent on $|\nabla \mathbf{p}(\mathbf{u}_0)|_{L^2}$ and T . By using the bound of $\mathcal{Y}(t)$ on the RHS of (2.20), we conclude that

$$\begin{cases} \partial_t u, \partial_t v, \sqrt{u+v} (|\partial_t u| + |\partial_t v|) \in L^2(L^2), \\ \nabla p_1, \nabla p_2 \in L^\infty(L^2) \quad (p_1, p_2 \in L^\infty(H^1)). \end{cases} \quad (2.21)$$

Noting that, by (1.7)-(1.9), $\nabla \mathbf{p} = \mathbf{P}(\mathbf{u}) \nabla \mathbf{u}$, we have

$$\nabla \mathbf{u} = \mathbf{P}(\mathbf{u})^{-1} \nabla \mathbf{p}. \quad (2.22)$$

Then in view of (2.6) and (2.21), we have

$$\nabla u, \nabla v \in L^\infty(L^2). \quad (2.23)$$

Remark 2.3. We deduce from (2.21)₂ that $\nabla p_i \in L^\infty(L^2)$ and we want to justify that $p_i \in L^\infty(H^1)$ through an equivalent norm. Indeed, we know that u, v are in $L^\infty(L^2)$ and thus $p_i \in L^\infty(L^1)$ (since $p_i \approx u^2 + v^2$). By the generalized Poincaré inequality (2.10) we see that $|\nabla \phi|_{L^2} + |\phi|_{L^1}$ is a norm on H^1 which is equivalent to the usual norm. This says that $p_i \in L^\infty(H^1)$.

From (2.14) we infer that

$$\mathbf{q}(\mathbf{u}), \ell(\mathbf{u}) \in L^2(0, T; L^2(\Omega)^2). \quad (2.24)$$

Then returning to (1.1) and using (2.21)₁, we see that

$$\Delta p_1, \Delta p_2 \in L^2(0, T; L^2(\Omega)), \quad (2.25)$$

and we have a priori bounds for $\Delta p_1, \Delta p_2$ in these spaces.

Lemma 2.2. *We assume that $\mathbf{u}_0 \in L^2(\Omega)^2$, $\mathbf{u}_0 \geq \mathbf{0}$ and $\nabla \mathbf{p}(\mathbf{u}_0) \in L^2(\Omega)^4$, that the coefficients a_{ij} satisfy (1.5), $d \leq 4$ and that $\mathbf{u} = (u, v)$ is a smooth ≥ 0 solution of (1.1), (1.2). Then u, v satisfy (2.21), (2.23)–(2.25). Furthermore the norms of \mathbf{u} and \mathbf{p} in these spaces can be bounded by constants depending on the norms of \mathbf{u}_0 and $\nabla \mathbf{p}(\mathbf{u}_0)$ in L^2 , on T and on the coefficients.*

2.3 Positivity and maximum principle

To deal with the positivity of the solutions, we will consider the following auxiliary systems

$$\partial_t \mathbf{u} - \nabla \cdot (\mathbf{P}(\mathbf{u}^+) \nabla \mathbf{u}) + \mathbf{q}(\mathbf{u}^+) = \ell(\mathbf{u}^+), \quad (2.26)$$

where $\mathbf{u}^+ = (u^+, v^+)$ and

$$\partial_t \mathbf{u} - \nabla \cdot (\mathbf{P}(\lambda_M(\mathbf{u})) \nabla \mathbf{u}) + \mathbf{q}(\lambda_M(\mathbf{u})) = \ell(\lambda_M(\mathbf{u})), \quad (2.27)$$

where $M > 0$ and

$$\lambda_M(\mathbf{u}) = (\lambda_M(u), \lambda_M(v)) \text{ with } \begin{cases} \lambda_M(u) = u^+ \text{ for } u \leq M \text{ and } M \text{ for } u > M, \\ \lambda_M(v) = v^+ \text{ for } v \leq M \text{ and } M \text{ for } v > M. \end{cases} \quad (2.28)$$

Note that if \mathbf{u} is a smooth solution of either system then $\mathbf{u} \geq \mathbf{0}$ ($u \geq 0, v \geq 0$) so that $\mathbf{u}^+ = \mathbf{u}$.

Indeed taking the scalar product in L^2 of (2.26) with $-\mathbf{u}^-$, we find

$$\begin{aligned} \bullet \quad & -\langle \partial_t \mathbf{u}, \mathbf{u}^- \rangle = \frac{1}{2} \frac{d}{dt} |\mathbf{u}^-|^2, \\ \bullet \quad & -\langle \nabla \cdot (\mathbf{P}(\mathbf{u}^+) \nabla \mathbf{u}), \nabla \mathbf{u}^- \rangle = -\langle \mathbf{P}(\mathbf{u}^+) \nabla \mathbf{u}, \nabla \mathbf{u}^- \rangle \\ & = -\int_{\Omega} \left[(d_1 + 2a_{11}u^+ + a_{12}v^+) \underbrace{\nabla u \nabla u^-}_{\leq 0} + a_{12}u^+ \underbrace{\nabla v \nabla u^-}_{=0} \right] dx \\ & \quad - \int_{\Omega} \left[a_{21} \underbrace{v^+ \nabla u \nabla v^-}_{=0} + (d_1 + a_{21}u^+ + 2a_{22}v^+) \underbrace{\nabla v \nabla v^-}_{\leq 0} \right] dx. \end{aligned}$$

With $\langle \nabla u, \nabla u^- \rangle = -|\nabla u^-|^2$, and the same for v , we see that $-\langle \nabla \cdot (\mathbf{P}(\mathbf{u}^+) \nabla \mathbf{u}), \mathbf{u}^- \rangle \geq 0$.

Similarly

$$\begin{aligned} \bullet \quad & -\int_{\Omega} [q_1(\mathbf{u}^+)u^- + q_2(\mathbf{u}^+)v^-] dx \\ & = \int_{\Omega} [(b_1u^+ + c_1v^+)u^+u^- + (b_2u^+ + c_2v^+)v^+v^-] dx = 0, \\ \bullet \quad & -\langle \ell_1(u^+), u^- \rangle - \langle \ell_2(v^+), v^- \rangle = -\int_{\Omega} [a_1u^+u^- + a_2v^+v^-] dx = 0. \end{aligned}$$

Finally

$$\frac{d}{dt} |\mathbf{u}^-|^2 \leq 0 \quad (2.29)$$

so that $\mathbf{u}^-(t) = 0$ for all time if $u_0 \geq 0, v_0 \geq 0$.

Then multiplying (2.27) by $-\mathbf{u}^-$, we find

$$\begin{aligned}
\bullet \quad & \langle \nabla \cdot (\mathbf{P}(\lambda_M(\mathbf{u})) \nabla \mathbf{u}), \mathbf{u}^- \rangle = - \langle (\mathbf{P}(\lambda_M(\mathbf{u})) \nabla \mathbf{u}), \nabla \mathbf{u}^- \rangle \\
& = - \int_{\Omega} \left[(d_1 + 2a_{11}\lambda_M(u) + a_{12}\lambda_M(v)) \underbrace{\nabla u \nabla u^-}_{\leq 0} + a_{12} \underbrace{\lambda_M(u) \nabla v \nabla u^-}_{=0} \right] \\
& \quad - \int_{\Omega} \left[a_{21} \underbrace{\lambda_M(v) \nabla u \nabla v^-}_{=0} + (d_1 + a_{21}\lambda_M(u) + 2a_{22}\lambda_M(v)) \underbrace{\nabla v \nabla v^-}_{\leq 0} \right]. \\
& \geq 0.
\end{aligned}$$

Also

$$\begin{aligned}
\bullet \quad & - \int_{\Omega} \left[q_1(\lambda_M(\mathbf{u}))u^- + q_2(\lambda_M(\mathbf{u}))v^- \right] dx \\
& = \int_{\Omega} \left[(b_1\lambda_M(u) + c_1\lambda_M(v))\lambda_M(u)u^- + (b_2\lambda_M(u) + c_2\lambda_M(v))\lambda_M(v)v^- \right] dx = 0, \\
\bullet \quad & - \langle \ell_1(\lambda_M(u)), u^- \rangle - \langle \ell_2(\lambda_M(v)), v^- \rangle = - \int_{\Omega} \left[a_1\lambda_M(u)u^- + a_2\lambda_M(v)v^- \right] dx = 0,
\end{aligned}$$

so that (2.29) holds again leading to $u(t) \geq 0, v(t) \geq 0$ for all t .

Orientation. It is not easy to take advantage of the above a priori estimates (in particular those of Section 2.1), and to construct positive solutions of the SKT equations satisfying these a priori estimates. We will deal with this issue in Section 3 by using finite differences in time, together with the truncation operator λ_M .

3 Finite differences in time

3.1 Finite differences for the truncated SKT equations

Let $T > 0$ be fixed. Consider two numbers $N, M > 0$ fixed for the moment but which will eventually go to infinity; N is an integer and $k = \Delta t = T/N$ is the time step.

We consider the finite difference scheme:

$$\left\{ \begin{array}{l} \frac{u_M^m - u_M^{m-1}}{k} - \nabla \cdot \left(p_{11}^M(u_M^m, v_M^m) \nabla u_M^n + p_{12}^M(u_M^m, v_M^m) \nabla v_M^n \right) \\ \quad + q_1^M(u_M^m, v_M^m) - \ell_1^M(u_M^m) = 0 \\ \frac{v_M^m - v_M^{m-1}}{k} - \nabla \cdot \left(p_{21}^M(u_M^m, v_M^m) \nabla u_M^n + p_{22}^M(u_M^m, v_M^m) \nabla v_M^n \right) \\ \quad + q_2^M(u_M^m, v_M^m) - \ell_2^M(v_M^m) = 0, \end{array} \right. \quad (3.1)$$

with boundary conditions

$$u_M^m = v_M^m = 0 \text{ on } \partial\Omega \quad (\text{or } \frac{\partial u_M^m}{\partial n} = \frac{\partial v_M^m}{\partial n} = 0 \text{ on } \partial\Omega)$$

and initial conditions

$$u_M^0 = u_0, v_M^0 = v_0.$$

We have set

$$p_{ij}^M(u_M^m, v_M^m) = p_{ij}(\lambda_M(u_M^m), \lambda_M(v_M^m)), \quad i, j = 1, 2. \quad (3.2a)$$

$$\ell_1^M(u_M^m) = \ell_1(\lambda_M(u_M^m)), \quad \ell_2^M(v_M^m) = \ell_1(\lambda_M(v_M^m)), \quad (3.2b)$$

However, for the q_i we write

$$\begin{cases} q_1^M(u_M^m, v_M^m) = (b_1 |u_M^m| + c_1 \lambda_M(v_M^m)) u_M^m, \\ q_2^M(u_M^m, v_M^m) = (b_2 \lambda_M(u_M^m) + c_2 |v_M^m|) v_M^m, \end{cases} \quad (3.2c)$$

Alternatively in variational form: $u_M^m, v_M^m \in V$ ($= H^1(\Omega)^2$ or $H_0^1(\Omega)^2$ depending on the b.c.) satisfying

$$\begin{cases} \langle u_M^m, \bar{u} \rangle + k \langle p_{11}^M(u_M^m, v_M^m) \nabla u_M^m, \nabla \bar{u} \rangle + k \langle p_{12}(u_M^m, v_M^m) \nabla v_M^m, \nabla \bar{u} \rangle \\ \quad + k \langle q_1^M(u_M^m, v_M^m), \bar{u} \rangle - k \langle \ell_1^M(u_M^m), \bar{u} \rangle = \langle u_M^{m-1}, \bar{u} \rangle, \\ \langle v_M^m, \bar{v} \rangle + k \langle p_{21}^M(u_M^m, v_M^m) \nabla u_M^m, \nabla \bar{v} \rangle + k \langle p_{22}(u_M^m, v_M^m) \nabla v_M^m, \nabla \bar{v} \rangle \\ \quad + k \langle q_2^M(u_M^m, v_M^m), \bar{v} \rangle - k \langle \ell_2^M(v_M^m), \bar{v} \rangle = \langle v_M^{m-1}, \bar{v} \rangle, \end{cases} \quad (3.3)$$

for every $\bar{u}, \bar{v} \in V$.

For the sake of simplicity we temporarily drop the lower index M and write u^m, v^m instead of u_M^m, v_M^m .

Because λ_M is a bounded function, the proof of existence of u^m, v^m follows very closely the proof of existence of solutions for the stationary Navier Stokes equations [Temam, 2001] after we notice the following “coercivity” properties obtained by setting $u = u^m$ and $v = v^m$ in the left hand sides of (3.3)₁ and (3.3)₂ and adding these equations

$$\begin{aligned} |u^m|^2 + k \langle p_{11}^M(u^m, v^m) \nabla u^m, \nabla u^m \rangle + k \langle p_{12}^M(u^m, v^m) \nabla v^m, \nabla u^m \rangle + k \langle q_1^M(u^m, v^m), u^m \rangle \\ - k \langle \ell_1^M(u^m), u^m \rangle + |v^m|^2 + k \langle p_{21}^M(u^m, v^m) \nabla u^m, \nabla v^m \rangle \\ + k \langle p_{22}^M(u^m, v^m) \nabla v^m, \nabla v^m \rangle + k \langle q_2^M(u^m, v^m), v^m \rangle \\ - k \langle \ell_2^M(v^m), v^m \rangle = \langle u^{m-1}, u^m \rangle - \langle v^{m-1}, v^m \rangle. \end{aligned} \quad (3.4)$$

Repeating the calculations in (2.3) under the assumptions in (1.5), we see that the sum of the p_{ij} terms is bounded from below by

$$\int_{\Omega} [d_0 + \alpha(\lambda_M(u^m) + \lambda_M(v^m))] [|\nabla u^m|^2 + |\nabla v^m|^2] dx. \quad (3.5)$$

The q_i terms give the lower bound

$$\begin{aligned} \int_{\Omega} [b_1 |u^m|^3 + c_1 \lambda_M(v^m) |u^m|^2 + b_2 \lambda_M(u^m) |v^m|^2 + c_2 |v^m|^3] dx \\ \geq \int_{\Omega} (b_1 |u^m|^3 + c_2 |v^m|^3) dx. \end{aligned} \quad (3.6)$$

We easily see that the ℓ_i terms are bounded from below by

$$-\int_{\Omega} \left[a_1 |u^m|^2 + a_2 |v^m|^2 \right] dx \geq -\frac{1}{2} \int_{\Omega} \left[b_1 |u^m|^3 + c_2 |v^m|^3 \right] dx - \mathcal{K}_1, \quad (3.7)$$

where \mathcal{K}_1 is an absolute constant.

Hence the expression (3.4) is bounded from below pointwise a.e. by

$$\begin{aligned} |u^m|^2 + |v^m|^2 + [d_0 + \alpha(\lambda_M(u^m) + \lambda_M(v^m))] \left(|\nabla u^m|^2 + |\nabla v^m|^2 \right) \\ + \frac{b_1}{2} |u^m|^3 + \frac{c_1}{2} |v^m|^3 - \mathcal{K}_1, \end{aligned} \quad (3.8)$$

which guarantees coercivity in (at least) V for (3.3).

As we said, by implementation of a Galerkin method, as for the stationary Navier-Stokes equations, we obtain the existence of $(u^m, v^m) \in V$ solutions of (3.3).

Then, as for (2.29), we show recursively, starting from $u_0, v_0 \geq 0$, that the u^m, v^m are ≥ 0 . Indeed, e.g. for u^m , we replace \bar{u} by $-(u^m)^- \in H^1(\Omega)$ (or $H_0^1(\Omega)$) in (3.3)₁. This gives

$$\begin{aligned} |(u^m)^-|^2 - k \int_{\Omega} \left(d_1 + 2a_{11}\lambda_M(u^m) + a_{12}\lambda_M(v^m) \right) \underbrace{\nabla u^m \nabla (u^m)^-}_{\leq 0} \\ - k \int_{\Omega} a_{12} \underbrace{\lambda_M(u^m) \nabla v^m \nabla (u^m)^-}_{=0} - \int_{\Omega} \left(b_1 |u^m| + c_1 \lambda_M(v^m) \right) \underbrace{u^m (u^m)^-}_{\leq 0} dx \\ + \int_{\Omega} a_1 \underbrace{\lambda_M(u^m) (u^m)^-}_{=0} dx = -\langle u^{m-1}, (u^m)^- \rangle. \end{aligned} \quad (3.9)$$

The RHS of (3.9) is ≤ 0 , since, by the induction assumption, $u^{m-1} \geq 0$. Hence

$$(u^m)^- = 0,$$

and $u^m = u_M^m \geq 0$. We proceed similarly for $v^m = v_M^m$.

Having shown that $u^m = u_M^m, v^m = v_M^m$ are ≥ 0 , we can drop the absolute values that we have introduced in the definition of q_1^M and q_2^m in (3.2c).

Remark 3.1. Now we remember that u^m, v^m actually depend on M , $u^m = u_M^m, v^m = v_M^m$, with $u_M^m, v_M^m \geq 0, u_M^m, v_M^m \in H^1(\Omega) \cap L^3(\Omega)$, and we want to let $M \rightarrow \infty$, to obtain a finite difference approximation for the SKT equations themselves.

3.2 Finite differences for the SKT equations

We now want to let $M \rightarrow \infty$ in (3.3) to obtain a solution to the finite difference scheme for the SKT equations themselves. For the moment M is still fixed.

Considering the solutions u_M^m, v_M^m of (3.3), we write $\bar{u} = 2u_M^m$ in (3.3)₁ and $\bar{v} = 2v_M^m$ in (3.3)₂ and add these equations. This gives

$$2\langle u_M^m - u_M^{m-1}, u_M^m \rangle + 2\langle v_M^m - v_M^{m-1}, v_M^m \rangle + 2\mathcal{L}_M^m = 0, \quad (3.10)$$

where \mathcal{L}_M^m is the expression in the left-hand-side of (3.4) (less $|u_M^m|^2 + |v_M^m|^2$).

With the classical relation $2\langle a - b, b \rangle = |a|^2 - |b|^2 + |a - b|^2$, (3.10) yields

$$|u_M^m|^2 + |v_M^m|^2 - |u_M^{m-1}|^2 - |v_M^{m-1}|^2 + |u_M^m - u_M^{m-1}|^2 + |v_M^m - v_M^{m-1}|^2 + 2\mathcal{L}_M^m = 0. \quad (3.11)$$

We then infer from (3.11) and the minorations (3.5) and (3.8) that

$$\begin{aligned} & |u_M^m|^2 + |v_M^m|^2 + |u_M^m - u_M^{m-1}|^2 + |v_M^m - v_M^{m-1}|^2 \\ & + 2k \int_{\Omega} \left(d_0 + \alpha(\lambda_M(u_M^m) + \lambda_M(v_M^m)) \right) \left(|\nabla u_M^m|^2 + |\nabla v_M^m|^2 \right) dx \\ & + kb_1 |u_M^m|_{L^3}^3 + kc_1 |v_M^m|_{L^3}^3 \leq \mathcal{K}_1 + |u_M^{m-1}|^2 + |v_M^{m-1}|^2, \end{aligned} \quad (3.12)$$

for $m = 1, \dots, N$. By addition and iteration we obtain that

$$|u_M^m|^2 + |v_M^m|^2 \leq \mathcal{K}_2, \quad (3.13a)$$

$$\sum_{m=1}^N |u_M^m - u_M^{m-1}|^2 + |v_M^m - v_M^{m-1}|^2 \leq \mathcal{K}_2, \quad (3.13b)$$

$$k \sum_{m=1}^N \left(|\nabla u_M^m|^2 + |\nabla v_M^m|^2 \right) \leq \mathcal{K}_2, \quad (3.13c)$$

$$k \sum_{m=1}^N \int_{\Omega} \left(\lambda_M(u_M^m) + \lambda_M(v_M^m) \right) \left(|\nabla u_M^m|^2 + |\nabla v_M^m|^2 \right) dx \leq \mathcal{K}_2, \quad (3.13d)$$

$$k \sum_{m=1}^N \left[|u_M^m|_{L^3}^3 + |v_M^m|_{L^3}^3 \right] \leq \mathcal{K}_2, \quad (3.13e)$$

where \mathcal{K}_2 is a constant independent of M and N (but which depends on $|u_0|^2 + |v_0|^2$ and the other data).

The other estimates will be used later on but for the moment, for N fixed, we infer from (3.13a) and (3.13c) that the u_M^m and v_M^m are bounded in $H^1(\Omega) \cap L^3(\Omega)$ independently of M , $m = 1, \dots, N$. Hence by a finite number of extraction of subsequence we see that for $M \rightarrow \infty$.

$$u_M^m \rightharpoonup u^m, \quad v_M^m \rightarrow v^m, \quad (3.14)$$

weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega) \cap L^3(\Omega)$ in dimensions $d \leq 4$ for some $u^m, v^m \in H^1(\Omega)$ which are ≥ 0 like the u_M^m, v_M^m . Also by additional extraction of subsequences, the convergences (3.14) hold almost everywhere in Ω .

It is relatively easy to pass to the limit in the relation (3.3), with $(\bar{u}, \bar{v}) \in V \cap \mathcal{C}(\bar{\Omega})^2$ using the convergences (3.14). We obtain that for each $m = 1, \dots, N$

$$\left\{ \begin{array}{l} \langle u^m, \bar{u} \rangle + k \langle p_{11}(u^m, v^m) \nabla u^m, \nabla \bar{u} \rangle + k \langle p_{12}(v^m, u^m) \nabla v^m, \nabla \bar{u} \rangle \\ \quad + k \langle q_1(u^m, v^m), \bar{u} \rangle - k \langle \ell_1(u^m), \bar{u} \rangle = \langle u^{m-1}, \bar{u} \rangle, \\ \langle v^m, \bar{v} \rangle + k \langle p_{21}(u^m, v^m) \nabla v^m, \nabla \bar{v} \rangle + k \langle p_{22}(v^m, u^m) \nabla v^m, \nabla \bar{v} \rangle \\ \quad + k \langle q_2(u^m, v^m), \bar{v} \rangle - k \langle \ell_2(v^m), \bar{v} \rangle = \langle v^{m-1}, \bar{v} \rangle, \end{array} \right. \quad (3.15)$$

for any $(\bar{u}, \bar{v}) \in V \cap \mathcal{C}(\bar{\Omega})^2$. We observe for this purpose that

$$\lambda_M(u_M^m) \rightarrow (u^m)^+ = u^m, \quad \lambda_M(v_M^m) \rightarrow (v^m)^+ = v^m. \quad (3.16)$$

a.e. and in $L^2(\Omega)$.

Additional information. Now we want to pay more attention to the terms $p_{ij}(u^m, v^m) \nabla u^m$, $p_{ij}(u^m, v^m) \nabla v^m$ and to show that (3.15) is valid in fact for any $\bar{u}, \bar{v} \in V$ (and not only $\bar{u}, \bar{v} \in V \cap \mathcal{C}(\bar{\Omega})^2$).

We start from the estimate (3.13d), and consider the expressions

$$\sqrt{\lambda_M(u_M^m)} \nabla u_M^m, \sqrt{\lambda_M(u_M^m)} \nabla v_M^m, \sqrt{\lambda_M(v_M^m)} \nabla u_M^m, \sqrt{\lambda_M(v_M^m)} \nabla v_M^m.$$

They are, each, bounded in $L^2(\Omega)$, and they contain each a subsequence which converges weakly in $L^2(\Omega)$ (m fixed, $M \rightarrow \infty$, finite extraction of subsequences). Due to (3.14) and the a.e. convergences, their respective limits are

$$\sqrt{(u^m)^+} \nabla u^m, \sqrt{(u^m)^+} \nabla v^m, \sqrt{(v^m)^+} \nabla u^m, \sqrt{(v^m)^+} \nabla v^m.$$

Passing to the lower limit in (3.13d), we see that

$$k \sum_{m=1}^N (u^m + v^m) \left(|\nabla u^m|^2 + |\nabla v^m|^2 \right) \leq \mathcal{K}_2. \quad (3.17)$$

This estimate will eventually lead to the results that $\sqrt{u} \nabla u, \sqrt{u} \nabla v, \sqrt{v} \nabla u, \sqrt{v} \nabla v$ belong to $L^2(0, T; L^2(\Omega))$ as $N \rightarrow \infty$.

For the moment, with N fixed, (3.16) and (3.13a) imply that $\sqrt{u^m} \nabla u^m$ and $\sqrt{v^m} \nabla v^m \in L^2(\Omega)$. Hence $(u^m)^{\frac{3}{2}}$ and $(v^m)^{\frac{3}{2}}$ belong to $H^1(\Omega)$ and by Sobolev inclusion, say in dimension $d = 4$ (the worse case) $(u^m)^{\frac{3}{2}}, (v^m)^{\frac{3}{2}} \in L^4(\Omega)$, that is $u^m, v^m \in L^6(\Omega)$. Hence the expressions $u^m \nabla u^m, u^m \nabla v^m, v^m \nabla u^m, v^m \nabla v^m$ all belong to $L^{\frac{12}{7}}(\Omega)$ as the product of an $L^2(\Omega)$ function with an $L^{\frac{12}{5}}(\Omega)$ function; e.g.

$$u^m \nabla u^m = \sqrt{u^m} \sqrt{u^m} \nabla u^m \in L^{\frac{12}{7}}(\Omega). \quad (3.18)$$

Hence the equations (3.15) are now valid for (\bar{u}, \bar{v}) belonging to V with $\nabla \bar{u}, \nabla \bar{v} \in L^{\frac{12}{5}}(\Omega)^2$.

More generally passing to the lower limit $M \rightarrow \infty$ in (3.13a)-(3.13e) we see that for $m = 0, \dots, N$

$$|u^m|^2 + |v^m|^2 \leq \mathcal{K}_2, \quad (3.19a)$$

$$\sum_{m=1}^N |u^m - u^{m-1}|^2 + |v^m - v^{m-1}|^2 \leq \mathcal{K}_2, \quad (3.19b)$$

$$k \sum_{m=1}^N \left(|\nabla u^m|^2 + |\nabla v^m|^2 \right) \leq \mathcal{K}_2, \quad (3.19c)$$

$$k \sum_{m=1}^N \int_{\Omega} (u^m + v^m) \left(|\nabla u^m|^2 + |\nabla v^m|^2 \right) dx \leq \mathcal{K}_2, \quad (3.19d)$$

$$k \sum_{m=1}^N |u^m|_{L^3}^3 + |v^m|_{L^3}^3 \leq \mathcal{K}_2, \quad (3.19e)$$

with the same constant \mathcal{K}_2 independent of N .

We conclude our a priori estimates for the finite solutions $\mathbf{u}^m = (u^m, v^m)$

Lemma 3.1. *Suppose that $\mathbf{u}^0 = \mathbf{u}_0 \in L^2(\Omega)^2$, $\mathbf{u}_0 \geq \mathbf{0}$ and that (1.5) holds. Then the bounds in (3.19) hold true for the solutions \mathbf{u}^m of (3.15).*

Remark 3.2. Later on we will extend this class of functions (\bar{u}, \bar{v}) such that (3.15) holds and study the dependance in t . For the moment we aim to derive for the u^m, v^m the discrete analogue of the a priori estimates (2.21).

3.3 More a priori estimates

Referring to the initial form (1.1) of the SKT equations, we rewrite (3.15) as

$$\begin{cases} \frac{u^m - u^{m-1}}{k} - \Delta p_1(u^m, v^m) + q_1(u^m, v^m) - \ell_1(u^m) = 0, \\ \frac{v^m - v^{m-1}}{k} - \Delta p_2(u^m, v^m) + q_2(u^m, v^m) - \ell_2(v^m) = 0, \end{cases} \quad (3.20)$$

with, as in (1.3),

$$\begin{cases} p_1(u^m, v^m) = (d_1 + a_{11}u^m + a_{12}v^m)u^m, \\ p_2(u^m, v^m) = (d_2 + a_{21}u^m + a_{22}v^m)v^m. \end{cases} \quad (3.21)$$

We have observed that, for fixed N , each u^m, v^m belongs to $L^6(\Omega)$ ($d \leq 4$). Hence $p_1(u^m, v^m)$, $p_2(u^m, v^m)$, $q_1(u^m, v^m)$, $q_2(u^m, v^m)$, $\ell_1(u^m)$, $\ell_2(v^m)$ belong to $L^3(\Omega)$. We also observe that $\nabla p_1(u^m, v^m) = (d_1 + 2a_{11}u^m + a_{12}v^m)\nabla u^m + a_{12}u^m\nabla v^m$, and, in view of (3.19c), (3.19d), and the similar relations, $\nabla p_1(u^m, v^m) \in L^{\frac{12}{7}}(\Omega)$, and the same is true for $\nabla p_2(u^m, v^m)$. Furthermore, $p_1^m = p_1(u^m, v^m)$, $p_2^m = p_2(u^m, v^m)$ satisfy the same boundary conditions as u^m, v^m , and hence, in view of (3.16), p_1^m, p_2^m belong to $W^{2,12/7}(\Omega)$ for each fixed m for N fixed. By bootstrapping, $\nabla p_1^m, \nabla p_2^m \in W^{1,12/7}(\Omega)^2 \subset L^3(\Omega)^2$ in space dimensions $d \leq 4$, so that $\nabla p_1^m, \nabla p_2^m \in L^2(\Omega)^2$, and $p_1^m, p_2^m \in H^2(\Omega)$, for $m = 0, 1, \dots, N$, N fixed.

We have the following a priori estimates

Lemma 3.2. Suppose that $\mathbf{u}^0 = \mathbf{u}_0 \in L^2(\Omega)^2$, $\mathbf{u}_0 \geq \mathbf{0}$, $\nabla \mathbf{p}(\mathbf{u}_0) \in L^2(\Omega)^4$ and that the conditions (1.5) are satisfied. Then the following bounds independent of N hold true

$$|\nabla \mathbf{p}(\mathbf{u}^m)|_{L^2(\Omega)^4} \leq c, m = 0, \dots, N, \quad (3.22a)$$

$$k \sum_{m=1}^N \int_{\Omega} \left[d_0 + \frac{\alpha}{4}(u^{m-1} + v^{m-1} + u^m + v^m) \right] \left| \frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{k} \right|^2 dx \leq c, \quad (3.22b)$$

where the constants c are independent of k .

Proof. We take the scalar product in $L^2(\Omega)$ of (3.20)₁ with $2k(p_1(\mathbf{u}^m) - p_1(\mathbf{u}^{m-1}))$ and (3.20)₂ by $2k(p_2(\mathbf{u}^m) - p_2(\mathbf{u}^{m-1}))$. It is clear that

$$-2k\langle \Delta p_1^m, p_1^m - p_1^{m-1} \rangle = k|\nabla p_1^m|^2 - k|\nabla p_1^{m-1}|^2 + k|\nabla(p_1^m - p_1^{m-1})|^2 \quad (3.23a)$$

and similarly

$$-2k\langle \Delta p_2^m, p_2^m - p_2^{m-1} \rangle = k|\nabla p_2^m|^2 - k|\nabla p_2^{m-1}|^2 + k|\nabla(p_2^m - p_2^{m-1})|^2. \quad (3.23b)$$

For the terms

$$2\langle (u^m - u^{m-1}), p_1^m - p_1^{m-1} \rangle + 2\langle (v^m - v^{m-1}), p_2^m - p_2^{m-1} \rangle, \quad (3.24)$$

we consider the mapping \mathcal{P} mentioned in Section 1:

$$\mathbf{u} = (u, v) \mapsto \mathbf{p} = (p_1, p_2) = \mathcal{P}(\mathbf{u}),$$

as defined by (1.8). The differential of \mathcal{P} is \mathbf{P} . Hence the term (3.24) can be seen as

$$2\langle \mathbf{u}^m - \mathbf{u}^{m-1}, \mathcal{P}(\mathbf{u}^m) - \mathcal{P}(\mathbf{u}^{m-1}) \rangle.$$

Now, we write:

$$\begin{aligned} \mathcal{P}(\mathbf{u}^m) - \mathcal{P}(\mathbf{u}^{m-1}) &= \int_0^1 \frac{d}{dt} \mathcal{P}(\mathbf{u}^{m-1} + t(\mathbf{u}^m - \mathbf{u}^{m-1})) dt \\ &= \int_0^1 \mathbf{P}((1-t)\mathbf{u}^{m-1} + t\mathbf{u}^m) \cdot (\mathbf{u}^m - \mathbf{u}^{m-1}) dt, \end{aligned}$$

and

$$\mathbf{p}^m - \mathbf{p}^{m-1} = \mathcal{P}(\mathbf{u}^m) - \mathcal{P}(\mathbf{u}^{m-1}) = \langle \overline{\mathbf{P}}^m(\mathbf{u}^m - \mathbf{u}^{m-1}), \mathbf{u}^m - \mathbf{u}^{m-1} \rangle, \quad (3.25)$$

with

$$\overline{\mathbf{P}}^m = \int_0^1 \mathbf{P}((1-t)\mathbf{u}^{m-1} + t\mathbf{u}^m) dt.$$

For $\mathbf{u}^{m-1} \geq \mathbf{0}$, $\mathbf{u}^m \geq \mathbf{0}$, $t \in [0, 1]$, we see that $(1-t)\mathbf{u}^{m-1} + t\mathbf{u}^m \geq \mathbf{0}$. Hence we can apply the bound (2.3) and we find

$$\begin{aligned} &\langle \mathbf{P}((1-t)\mathbf{u}^{m-1} + t\mathbf{u}^m) \cdot (\mathbf{u}^m - \mathbf{u}^{m-1}), \mathbf{u}^m - \mathbf{u}^{m-1} \rangle \\ &\geq \alpha((1-t)(u^{m-1} + v^{m-1}) + t(u^m + v^m)) |\mathbf{u}^m - \mathbf{u}^{m-1}|^2 + d_0 |\mathbf{u}^m - \mathbf{u}^{m-1}|^2, \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \langle \mathbf{P}((1-t)\mathbf{u}^{m-1} + t\mathbf{u}^m) \cdot (\mathbf{u}^m - \mathbf{u}^{m-1}), \mathbf{u}^m - \mathbf{u}^{m-1} \rangle dt \\ \geq \left[d_0 + \frac{\alpha}{2}(u^{m-1} + v^{m-1} + u^m + v^m) \right] |\mathbf{u}^m - \mathbf{u}^{m-1}|^2. \end{aligned}$$

Finally (3.24) is bounded from below by

$$\left[2d_0 + \alpha(u^{m-1} + v^{m-1} + u^m + v^m) \right] |\mathbf{u}^m - \mathbf{u}^{m-1}|^2. \quad (3.26)$$

Shifting the terms $q_1 - \ell_1$ and $q_2 - \ell_2$ to the right-hand side of (3.15)₁ and (3.15)₂, we now look for an upper bound of

$$-2k \langle \mathbf{q}(\mathbf{u}^m) - \ell(\mathbf{u}^m), \mathbf{p}^m - \mathbf{p}^{m-1} \rangle. \quad (3.27)$$

Using (3.25), we bound the expression (3.27) from above by

$$\begin{aligned} -2k \langle \mathbf{q}(\mathbf{u}^m) - \ell(\mathbf{u}^m), \bar{\mathbf{P}}^m(\mathbf{u}^m - \mathbf{u}^{m-1}) \rangle \\ \leq \frac{\alpha}{4}(u^{m-1} + v^{m-1} + u^m + v^m) |\mathbf{u}^m - \mathbf{u}^{m-1}|^2 + ck^2 \left(|\mathbf{u}^{m-1}|^5 + |\mathbf{u}^m|^5 \right). \end{aligned}$$

We arrive at

$$\begin{aligned} \int_{\Omega} \left[d_0 + \frac{\alpha}{4}(u^{m-1} + v^{m-1} + u^m + v^m) \right] |\mathbf{u}^m - \mathbf{u}^{m-1}|^2 dx \\ + k |\nabla \mathbf{p}^m|_{L^2}^2 - k |\nabla \mathbf{p}^{m-1}|_{L^2}^2 + k |\nabla(\mathbf{p}^m - \mathbf{p}^{m-1})|_{L^2}^2 \\ \leq ck^2 \left(|\mathbf{u}^{m-1}|_{L^5}^5 + |\mathbf{u}^m|_{L^5}^5 \right). \quad (3.28) \end{aligned}$$

Using exactly the same calculations as for the bounds (2.19a)–(2.19b), we find

$$|\mathbf{u}^{m-1}|_{L^5}^5 + |\mathbf{u}^m|_{L^5}^5 \leq c(|\nabla \mathbf{p}^{m-1}|_{L^2}^2 + |\nabla \mathbf{p}^m|_{L^2}^2 + 1).$$

Thus after dividing (3.28) by k , we obtain

$$\begin{aligned} k \int_{\Omega} \left[d_0 + \frac{\alpha}{4}(u^{m-1} + v^{m-1} + u^m + v^m) \right] \left| \frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{k} \right|^2 dx \\ + |\nabla \mathbf{p}^m|_{L^2}^2 - |\nabla \mathbf{p}^{m-1}|_{L^2}^2 + |\nabla(\mathbf{p}^m - \mathbf{p}^{m-1})|_{L^2}^2 \\ \leq ck(|\nabla \mathbf{p}^{m-1}|_{L^2}^2 + |\nabla \mathbf{p}^m|_{L^2}^2 + 1). \quad (3.29) \end{aligned}$$

The inequality (3.29) in particular implies

$$\frac{|\nabla \mathbf{p}^m|_{L^2}^2 - |\nabla \mathbf{p}^{m-1}|_{L^2}^2}{k} \leq c \left(\frac{|\nabla \mathbf{p}^m|_{L^2}^2 + |\nabla \mathbf{p}^{m-1}|_{L^2}^2}{2} + 1 \right). \quad (3.30)$$

We now apply the discrete Gronwall inequality A.1 for $a_m = |\nabla \mathbf{p}^m|_{L^2}^2$, $\tau_m = k$, $\theta = \frac{1}{2}$, $\lambda_m = g_m = c$. We note that $\omega_\ell = (1 + \frac{k}{2})/(1 - \frac{k}{2}) = 1 + \frac{2k}{2-k}$ and hence $\prod \omega_\ell \leq e^T$; from this, we are able to show that $|\nabla \mathbf{p}^m|_{L^2}^2$ is bounded uniformly for $m = 1, \dots, N$ by a constant depending on $|\nabla \mathbf{p}(\mathbf{u}_0)|_{L^2}$ and T . In other words, we have established the a priori estimate (3.22a).

By using the bound (3.22a) in the RHS of (3.30), we also obtain (3.22b).

Lemma 3.2 is proven. \square

3.4 Further a priori estimates

Similar to the estimates in (2.14), (2.24) and (2.25), we also have the following a priori estimates as consequences of the bounds in (3.19), (3.22):

Lemma 3.3. *With the same assumptions as in Lemma 3.2, we have*

$$k \sum_{m=1}^N |\mathbf{q}(\mathbf{u}^m)|_{L^2}^2, \quad k \sum_{m=1}^N |\ell(\mathbf{u}^m)|_{L^2}^2 \leq c, \quad (3.31a)$$

$$k \sum_{m=1}^N |\Delta \mathbf{p}(\mathbf{u}^m)|_{L^2}^2 \leq c, \quad (3.31b)$$

$$|\nabla \mathbf{u}^m|_{L^2} \leq c \text{ for } m = 0, 1, \dots, N. \quad (3.31c)$$

for c depending on T and $|\mathbf{p}(\mathbf{u}_0)|_{L^2}$ but not on N (nor k).

Proof. Here (3.31a) is a consequence of (3.19a), (3.19d) and (3.19e). In fact,

$$\begin{aligned} k \sum_{m=1}^N |u^m|_{L^4}^4 &\leq (\text{Hölder inequality}) \leq k \sum_{m=1}^N |u^m|_{L^2} |u^m|_{L^6}^3 = k \sum_{m=1}^N |u^m|_{L^2} \left| (u^m)^{\frac{3}{2}} \right|_{L^4}^2 \\ &\leq (\text{using (3.19a)}) \leq \mathcal{K}_2 k \sum_{m=1}^N \left| (u^m)^{\frac{3}{2}} \right|_{L^4}^2 \leq (\text{Sobolev embedding for } d = 4) \\ &\leq C k \sum_{m=1}^N \|(u^m)^{\frac{3}{2}}\|_{H^1}^2 \leq (\text{using (3.19d) \& (3.19e)}) \leq C, \end{aligned}$$

and together with the similar bound for $k \sum_{m=1}^N |v^m|_{L^4}^4$, we find the bound for $k \sum_{m=1}^N |\mathbf{q}(\mathbf{u}^m)|_{L^2}^2$.

Next, the bound (3.31b) is easily obtained by using the system (3.20), the bound of the term $d_0(\mathbf{u}^m - \mathbf{u}^{m-1})/k$ in (3.22b) and the bounds of $\mathbf{q}(\mathbf{u}^m), \ell(\mathbf{u}^m)$ in (3.31a).

Finally, we infer (3.31c) from the estimate (3.22a) and the relations (2.6), and (2.22).

□

3.5 Passage to the limit

In this section, we pass to the limit of the system (3.15), or equivalently (3.20). We first introduce the finite difference approximate functions. For each fixed time step k we associate to the finite difference solutions $\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^N$, the approximate functions $\mathbf{u}_k = (u_k, v_k)$, $\tilde{\mathbf{u}}_k = (\tilde{u}_k, \tilde{v}_k)$, as follows:

- $\mathbf{u}_k(t) = \mathbf{u}^m$, $t \in [(m-1)k, mk]$, $m = 1, \dots, N$.

- $\tilde{\mathbf{u}}_k(t)$ is the continuous function linear on each time interval $[(m-1)k, mk]$ and equal to \mathbf{u}^m at $t = mk$, $m = 0, 1, \dots, N$.

The finite difference system (3.15) is written in terms of $\mathbf{u}_k, \tilde{\mathbf{u}}_k$ as follows

$$\begin{cases} \langle \partial_t \tilde{\mathbf{u}}_k, \bar{\mathbf{u}} \rangle + \langle \nabla \mathbf{p}(\mathbf{u}_k), \nabla \bar{\mathbf{u}} \rangle + \langle \mathbf{q}(\mathbf{u}_k), \bar{\mathbf{u}} \rangle = \langle \ell(\mathbf{u}_k), \bar{\mathbf{u}} \rangle, \\ \mathbf{u}_k(0) = \mathbf{u}_0 \text{ in } \Omega, \end{cases} \quad (3.32)$$

for all $\bar{\mathbf{u}} \in V$.

Assuming only that $\mathbf{u}_0 \in L^2(\Omega)^2$ (and $\mathbf{u}_0 \geq 0$) we infer from (3.19), (3.22) and (3.31) that $\mathbf{u}_k, \tilde{\mathbf{u}}_k$, are bounded independently of k as follows

$$\bullet \|\tilde{\mathbf{u}}_k - \mathbf{u}_k\|_{L^2(0,T;L^2)}^2 = \sum_{m=1}^N \left| \frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{k} \right|^2 \int_{t_{m-1}}^{t_m} (t - t_m)^2 dt \leq \frac{\mathcal{K}_2}{3} k, \quad (3.33a)$$

$$\bullet \mathbf{u}_k, \mathbf{u}_k^{\frac{3}{2}} \text{ belong to a bounded set in } L^2(0, T; H^1(\Omega)^2), \quad (3.33b)$$

$$\bullet \tilde{\mathbf{u}}_k, \tilde{\mathbf{u}}_k^{\frac{3}{2}} \text{ belong to a bounded set in } L^2(\eta, T; H^1(\Omega)^2), \forall \eta > 0, \quad (3.33c)$$

$$\bullet \mathbf{u}_k, \tilde{\mathbf{u}}_k \text{ belong to a bounded set in } L^\infty(0, T; L^2(\Omega)^2). \quad (3.33d)$$

Note that we do not use Lemma 3.2 at this stage because we only assume that $\mathbf{u}_0 \in L^2(\Omega)^2$ (and $\mathbf{u}_0 \geq 0$).

We infer from the above estimates that there exists a subsequence still denoted $k \rightarrow 0$, and $\mathbf{u}, \tilde{\mathbf{u}} \in L^\infty(0, T; L^2(\Omega)^2) \cap L^2(0, T; H^1(\Omega)^2)$, such that

- $\mathbf{u}_k \rightarrow \mathbf{u}$ in $L^\infty(0, T; L^2(\Omega)^2)$ weak* and $L^2(0, T; H^1(\Omega)^2)$ weakly,
- $\tilde{\mathbf{u}}_k \rightarrow \tilde{\mathbf{u}}$ in $L^\infty(0, T; L^2(\Omega)^2)$ weak* and $L^2(0, T; H^1(\Omega)^2)$ weakly,

and $\mathbf{u} = \tilde{\mathbf{u}}$. In order to conclude that $\mathbf{u}_k^{\frac{3}{2}}$ (and $\tilde{\mathbf{u}}_k^{\frac{3}{2}}$) converges to $\mathbf{u}^{\frac{3}{2}}$ and that $\mathbf{p}(\mathbf{u}_k)$ converges to $\mathbf{p}(\mathbf{u})$, we proceed by compactness and derive a strong convergence result.

We rewrite (3.20)₁ in the form

$$\partial_t \tilde{u}_k - \Delta p_1(\mathbf{u}_k) + q_1(\mathbf{u}_k) - \ell_1(u_k) = 0. \quad (3.34)$$

We know that \mathbf{u}_k is bounded in $L^4(0, T; L^4(\Omega)^2)$ and $L^\infty(0, T; L^2(\Omega)^2)$ so that $q_1(\mathbf{u}_k), \ell_1(u_k)$ are both bounded in $L^2(0, T; L^2(\Omega))$. The term $\Delta p_1(\mathbf{u}_k)$ is written as

$$\Delta p_1(\mathbf{u}_k) = \nabla \cdot [(d_1 + 2a_{11}u_k + a_{12}v_k)\nabla u_k + a_{12}u_k\nabla v_k].$$

Considering the typical term $u_k\nabla u_k$, we write

$$\int_{\Omega} |u_k \nabla u_k|^{\frac{4}{3}} \leq \left(\int_{\Omega} u_k^4 \right)^{\frac{1}{3}} \left(\int_{\Omega} |\nabla u_k|^2 \right)^{\frac{2}{3}} = |\nabla u_k|_{L^2}^{\frac{4}{3}} |u_k|_{L^4}^{\frac{4}{3}},$$

and this product belongs to L_t^1 because the first function belongs to $L_t^{\frac{3}{2}}$ and the second one belongs to L_t^3 .

From this we conclude that $\partial_t \tilde{u}_k$ belongs to a bounded set of $L^{\frac{4}{3}}(0, T; W^{-1, \frac{4}{3}}(\Omega))$ ($W^{-1, \frac{4}{3}} = \nabla L^{\frac{4}{3}}$ = dual of $W_0^{1,4}$)¹. Using Aubin's compactness theorem A.2, we conclude that u_k and \tilde{u}_k converge to

¹see e.g. [Lions, 1965, Definition 5.1] for the definition of the space $W^{-1,p}(\Omega)$.

u in $L^2(0, T; L^2(\Omega))$ strongly and by an additional extraction of subsequence, that u_k converges to u a.e. in $(0, T) \times \Omega$. Then, by a standard argument see [Lions, 1969, Lemma 1.3, Ch. 1], we conclude that

$$u_k^{\frac{3}{2}} \rightharpoonup u^{\frac{3}{2}} \text{ and } p_1(\mathbf{u}_k) \rightharpoonup p_1(\mathbf{u}) \text{ weakly in } L^2(0, T; L^2(\Omega)). \quad (3.35)$$

With the same reasoning for v_k , we conclude that $\mathbf{u} = (u, v)$ satisfy (1.1) or (1.4). The initial and boundary condition (1.2) are proven in a classical way (in a weak/variational form in the case of the Neumann boundary condition).

If in addition, we assume that $\nabla \mathbf{p}(\mathbf{u}_0) \in L^2(\Omega)^4$, then the estimates of Lemma 3.2 imply by an additional extraction of subsequence that

$$\nabla \mathbf{p}(\mathbf{u}) \in L^\infty(0, T; L^2(\Omega)^4), \quad (1 + |u| + |v|)^{\frac{1}{2}} (|\partial_t u| + |\partial_t v|) \in L^2(0, T; L^2(\Omega)), \quad (3.36)$$

and returning to equations (1.1), we see that

$$\Delta \mathbf{p}(\mathbf{u}) \in L^2(0, T; L^2(\Omega)^2). \quad (3.37)$$

In summary we have proven the following

Theorem 3.1 (Existence of solutions).

- i) We assume that that $d \leq 4$, that the condition (1.5) hold, and that \mathbf{u}_0 is given, $\mathbf{u}_0 \in L^2(\Omega)^2$, $\mathbf{u}_0 \geq 0$. Then equation (1.1) possesses a solution $\mathbf{u} \geq 0$ such that, for every $T > 0$:

$$\mathbf{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)^2) \quad (3.38a)$$

$$(\sqrt{u} + \sqrt{v})(|\nabla u| + |\nabla v|) \in L^2(0, T; L^2(\Omega)) \quad (3.38b)$$

$$\mathbf{u} \in L^4(0, T; L^4(\Omega)). \quad (3.38c)$$

with the norms in these spaces bounded by a constant depending boundedly on T , on the coefficients, and on the norms in $L^2(\Omega)$ of u_0 and v_0 .

- ii) If, in addition, $\nabla \mathbf{p}(\mathbf{u}_0) \in L^2(\Omega)^4$, then the solution \mathbf{u} also satisfies (3.36) and (3.37), with the norms in these spaces bounded by a constant depending boundedly on the norms of \mathbf{u}_0 and $\nabla \mathbf{p}(\mathbf{u}_0)$ in L^2 (and on T and the coefficients).

4 Attractor

Since the uniqueness of weak solutions to the SKT equation is not available, we will develop a concept of weak attractor similar to what has been done for the three-dimensional Navier-Stokes equations in [Ball, 1997], [Sell, 1996], [Foias and Temam, 1987], [Foias et al., 2001], [Foias et al., 2010].

We follow closely [Foias et al., 2001] (see chapter III Section 4 and Appendix A5). The steps of the proof are as follows:

- We define (make more precise) the concept of weak solutions

- We derive time uniform estimates valid on $[0, \infty]$ and prove the existence of an absorbing set
- We define the weak attractor \mathcal{A}_w , show that it is compact and that it attacks all trajectories (in a sense to be specified).

4.1 Weak solutions of the SKT equations

Let $H = L^2(\Omega)^2$ and let H_w be the space H endowed with the weak topology. Here we call *weak* solution of the SKT equation any function \mathbf{u} satisfying (3.38) and (1.1), (1.2) in weak (variational) form in the case of the Neumann boundary condition. We require in addition that it satisfies the following energy inequality for all $t \geq 0$:

$$\begin{aligned} & \frac{1}{2} |u(t)|^2 + \frac{1}{2} |v(t)|^2 \\ & + \int_0^t \int_{\Omega} [p_{11}(\mathbf{u})(\nabla u)^2 + p_{12}(\mathbf{u})\nabla u \nabla v + p_{21}(\mathbf{u})\nabla u \nabla v + p_{22}(\mathbf{u})(\nabla v)^2] dx ds \\ & + \int_0^t \int_{\Omega} [q_1(\mathbf{u})u + \ell_1(\mathbf{u})u + q_2(\mathbf{u})u + \ell_2(\mathbf{u})v] dx ds \leq \frac{1}{2}(|u_0|^2 + |v_0|^2). \end{aligned} \quad (4.1)$$

Note that these inequalities are satisfied by the solutions \mathbf{u} provided by Theorem 3.1. Indeed, we go back to (3.4) and add this equation to the similar equation for v and then pass to the lower limit $M \rightarrow \infty$. Then we reinterpret the inequality that we obtain in terms of \mathbf{u}_k and pass again to the lower limit $k \rightarrow 0$. For this last passage to the limit we observe that

$$\int_0^t \int_{\Omega} q_1(\mathbf{u}_k)u_k dx ds \rightarrow \int_0^t \int_{\Omega} q_1(\mathbf{u})u dx ds, \quad (4.2a)$$

$$\int_0^t \int_{\Omega} \ell_1(\mathbf{u}_k)u_k dx ds \rightarrow \int_0^t \int_{\Omega} \ell_1(\mathbf{u})u dx ds, \quad (4.2b)$$

because \mathbf{u}_k is bounded in $L^4(0, T; L^4(\Omega)^2)$ and $\mathbf{u}_k \rightarrow \mathbf{u}$ a.e. in $\Omega \times (0, T)$, using again ([Lions, 1969, Lemma 1.3, Ch. 1]) and the same for the terms corresponding to q_2 and ℓ_2 . For the other terms, we pass to the lower limit:

$$\liminf_{k \rightarrow 0} |u_k(t)|^2 \geq |u(t)|^2, \quad \liminf_{k \rightarrow 0} |v_k(t)|^2 \geq |v(t)|^2, \quad (4.3)$$

$$\liminf_{k \rightarrow 0} \int_0^t \int_{\Omega} (\mathbf{P}(\mathbf{u}_k) \nabla \mathbf{u}_k) \nabla \mathbf{u}_k dx ds \geq \int_0^t \int_{\Omega} (\mathbf{P}(\mathbf{u}) \nabla \mathbf{u}) \nabla \mathbf{u} dx ds. \quad (4.4)$$

For (4.4), we use the fact that \mathbf{P} is positive definite, see (2.5).

Remark 4.1.

- Observe as for $\partial_t \tilde{\mathbf{u}}_k$ above, that if \mathbf{u} is a weak solution of the SKT equations on $(0, T)$, then $\partial_t \mathbf{u} \in L^{\frac{4}{3}}(0, T; W^{-1, \frac{4}{3}}(\Omega)^2)$ so that \mathbf{u} is continuous from $[0, T]$ into $W^{-1, \frac{4}{3}}(\Omega)^2$ (after modification on a set of measure 0), and since \mathbf{u} belongs to $L^\infty(0, T; H)$, \mathbf{u} is also continuous from $[0, T]$ into H_w :

$$\partial_t \mathbf{u} \in L^{\frac{4}{3}}(0, T; W^{-1, \frac{4}{3}}(\Omega)^2), \quad \mathbf{u} \in \mathcal{C}([0, T]; W^{-1, \frac{4}{3}}(\Omega)^2) \cap \mathcal{C}([0, T]; H_w). \quad (4.5)$$

- ii) We observe as in [Foias et al., 2001] that (4.1) also holds between two times t_1 and t , instead of 0 and t , with $0 \leq t_1 < t$ for all t' 's and for all t'_1 's in a dense subset of $(0, t)$ of total measure. From this we deduce that

$$\frac{d}{dt}\mathcal{Y}(t) + \int_{\Omega} (\mathbf{P}(\mathbf{u})\nabla\mathbf{u})\nabla\mathbf{u} \, dx + \int_{\Omega} (\mathbf{q}(\mathbf{u})\mathbf{u} - \ell(\mathbf{u})\mathbf{u}) \, dx \leq 0 \quad (4.6)$$

where $\mathcal{Y}(t) = |u(t)|^2 + |v(t)|^2$. Same proof as for (7.5) and (7.7) in [Foias et al., 2001, Chap. II].

- iii) Concatenation: we observe that if \mathbf{u}^1 is a weak solution of the SKT system on $(0, t_1)$ in the sense given above, and if \mathbf{u}^2 is a weak solution on $(0, t_2)$ with $\mathbf{u}^2(0) = \mathbf{u}^2(t_1)$, then the function \mathbf{u} equal to \mathbf{u}^1 on $(0, t_1)$ and to $\mathbf{u}^2(t - t_1)$ on $(t_1, t_2 - t_1)$, is a weak solution on $(0, t_1 + t_2)$; see [Foias et al., 2001].

4.2 Absorbing set

We now want to derive time uniform estimates on $(0, \infty)$ and prove the existence of an absorbing set in H .

With $\mathbf{P} \geq 0$, and $\mathbf{u} \geq \mathbf{0}$, we infer from (4.6) that

$$\frac{d}{dt}\mathcal{Y} + 2 \int_{\Omega} (b_1 u^3 + c_2 v^3) \, dx \leq 2 \int_{\Omega} (a_1 u^2 + a_2 v^2) \, dx, \quad (4.7)$$

with again $\mathcal{Y}(t) = |u(t)|^2 + |v(t)|^2$. With two (four) utilizations of Young's inequality, we infer from (4.7) that

$$\mathcal{Y}(t)' + \alpha_1 \mathcal{Y} \leq \alpha_2, \quad (4.8)$$

where α_1, α_2 are absolute constants.

Gronwall's lemma then implies that

$$\mathcal{Y}(t) \leq \mathcal{Y}(0)e^{-\alpha_1 t} + \frac{\alpha_2}{\alpha_1}(1 - e^{-\alpha_1 t}), \quad \forall t \geq 0. \quad (4.9)$$

This shows that $\mathcal{Y}(t) = |u(t)|^2 + |v(t)|^2$ is uniformly bounded for $t \geq 0$:

$$\mathcal{Y}(t) \leq \mathcal{Y}(0) + \frac{\alpha_2}{\alpha_1}, \quad (4.10)$$

and that, as $t \rightarrow \infty$

$$\limsup_{t \rightarrow \infty} \mathcal{Y}(t) \leq \frac{\alpha_2}{\alpha_1}. \quad (4.11)$$

From this we deduce that the ball of H , $B_{2\frac{\alpha_2}{\alpha_1}}(\mathbf{0})$, centered at $\mathbf{0}$ of radius $2\frac{\alpha_2}{\alpha_1}$ (or $r\frac{\alpha_2}{\alpha_1}$, $\forall r > 1$) is an *absorbing ball* in H for the SKT system.

4.3 The weak global attractor

We now define the weak global attractor of the SKT equations as the set \mathcal{A}_w of points φ in H , $\varphi \geq \mathbf{0}$, which belong to a complete trajectory, $\mathbf{u} \geq \mathbf{0}$, that is a weak solution on \mathbb{R} (or on (s, ∞) , $\forall s \in \mathbb{R}$) of the SKT equations, with $\mathbf{u} \geq \mathbf{0}$. We will show that \mathcal{A}_w is non-empty, compact in H_w invariant by the flow and that it attracts all weak solutions in H_w . Invariant by the flow means here that any trajectory, that is a weak solution of (1.1) and starts from a point $\mathbf{u}_0 \in \mathcal{A}_w$ is entirely included in \mathcal{A}_w .

The set \mathcal{A}_w is not empty since it contains the point $\mathbf{0} = (0, 0)$. Let us show that \mathcal{A}_w is compact in H_w . It follows clearly from (4.9)–(4.11) that

$$\mathcal{A}_w \subset B_{\frac{\alpha_2}{\alpha_1}}(\mathbf{0}), \quad (4.12)$$

\mathcal{A}_w is included in the ball of H centered at $\mathbf{0}$ of radius α_2/α_1 . If we show that \mathcal{A}_w is closed in H_w , then we will conclude that

$$\mathcal{A}_w \text{ is compact in } H_w. \quad (4.13)$$

Let φ_j be a sequence of \mathcal{A}_w . Each φ_j belongs to a complete trajectory \mathbf{u}^j (weak solution of the SKT equations on all of \mathbb{R}), and say $\varphi_j = \mathbf{u}^j(0)$. The sequence \mathbf{u}^j is bounded in H by α_2/α_1 , according to (4.12). We consider a sequence $t_j \rightarrow \infty$ and hence $\mathbf{u}^j(-t_j)$ is bounded in H , and we can show that the norms of \mathbf{u}^j appearing in (3.38) are bounded on $(-t_j, T)$ by a constant depending on T but not on j . Hence we can extract from \mathbf{u}^j a sequence still denoted \mathbf{u}^j which converges to a limit \mathbf{u} on $(-s, T)$, $\forall s, T > 0$, in the sense of (3.33a)–(3.33d), and \mathbf{u} is a weak solution of the SKT equations on $(-s, T)$, $\forall s, T > 0$ that is \mathbf{u} is a complete trajectory. Also $\varphi_j = \mathbf{u}^j(0)$ converges weakly in H to $\varphi = \mathbf{u}(0)$ so that $\varphi \in \mathcal{A}_w$ and \mathcal{A}_w is closed in H_w .

The invariance of \mathcal{A}_w follows from the concatenation property mentioned in Remark 4.1, iii). Consider a weak solution \mathbf{u} starting from $\mathbf{u}_0 = \mathbf{u}(0) \in \mathcal{A}_w$. Since $\mathbf{u}_0 \in \mathcal{A}_w$ it belongs to a complete trajectory $\tilde{\mathbf{u}}$ with say $\tilde{\mathbf{u}}(0) = \mathbf{u}_0$. By the concatenation property mentioned in Remark 4.1, iii) the function \mathbf{u}^* equal to $\tilde{\mathbf{u}}$ for $t \leq 0$ and to \mathbf{u} for $t \geq 0$ is a complete trajectory and it is therefore included in \mathcal{A}_w .

There remains to show that \mathcal{A}_w attracts all weak solutions in H_w as $t \rightarrow \infty$. We proceed as in [Foias et al., 2001]. We will show a stronger result, namely that \mathcal{A}_w attracts all the solutions in the weak topology of H , *uniformly* for all the initial conditions in a bounded set of H . Indeed, consider a sequence of initial data \mathbf{u}_{0n} bounded in H (with $\mathbf{u}_{0n} \geq \mathbf{0}$):

$$|\mathbf{u}_{0n}| \leq c_1 \text{ for all } n \in \mathbb{N}, \quad (4.14)$$

and consider the corresponding weak solutions provided by Theorem 3.1 i), $(\mathbf{u}_n)_{n \in \mathbb{N}}$. Consider a neighborhood \mathcal{U} of \mathcal{A}_w in H_w . We will show that there exists $t_1 = t_1(\mathcal{U})$, such that $\mathbf{u}_n(t) \in \mathcal{U}$, $\forall t \geq t_1$ and $\forall n \in \mathbb{N}$.

The proof of this result is by contradiction. Assume the property is not true: then there exists a neighborhood \mathcal{U} of \mathcal{A}_w in H_w and two sequences $\{n_j\}_j, n_j \in \mathbb{N}$ and $\{t_j\}_j, t_j \rightarrow \infty$ such that $\mathbf{u}_{n_j}(t_j)$ does not belong to \mathcal{U} . We deduce from (4.14) and (4.10) that the sequence $\mathbf{u}_{n_j}(t_j)$ is bounded in

H . Hence extracting a subsequence from $\{n_j\}_j$ and $\{t_j\}_j$ (still denoted by j), there exists $\mathbf{v}_0 \in H$ with of course $\mathbf{v}_0 \geq \mathbf{0}$, such that

$$\mathbf{u}_{n_j}(t_j) \rightharpoonup \mathbf{v}_0 \text{ weakly in } H. \quad (4.15)$$

Since $\mathbf{u}_{n_j}(t_j) \notin \mathcal{U}$ and since \mathcal{U} is a neighborhood of \mathcal{A}_w , it follows that $\mathbf{v}_0 \notin \mathcal{A}_w$. We now show that \mathbf{v}_0 belongs to a complete trajectory, so that $\mathbf{v}_0 \in \mathcal{A}_w$ thus establishing a contradiction. Indeed, define $\mathbf{v}_j(t) = \mathbf{u}_{n_j}(t + t_j)$, for $t \geq -t_j$. It is clear that \mathbf{v}_j is a weak solution of (1.1) on $(-t_j, \infty)$, with $\mathbf{v}_j(0) = \mathbf{u}_{n_j}(t_j)$. The sequence \mathbf{v}_j is bounded in $L^\infty(-t_j, \infty; H)$ and satisfies a priori estimates similar to (3.38a-c), and these a priori estimates are independant of j because $\mathbf{v}_j(0) = \mathbf{u}_{n_j}(t_j)$ is bounded in H . We deduce that there exists a subsequence, still denoted by j , which converges to \mathbf{v} on any interval $(-s, T)$ in the sense of (3.38a-c), and $\mathbf{v} \geq \mathbf{0}$ is a complete trajectory and $\mathbf{v}(0) = \lim \mathbf{v}_j(0) = \mathbf{v}_0$ by (4.15) (and $\mathbf{v}_j(0) = \mathbf{u}_{n_j}(t_j)$). The contradiction is established and we have shown that \mathcal{A}_w attracts all trajectories.

In summary we have proven the following

Theorem 4.1. *Assume that (1.5) holds. Then the set \mathcal{A}_w of all solutions $\mathbf{u} \geq \mathbf{0}$ of (1.1) on all of \mathbb{R} belonging to $L^\infty(\mathbb{R}; H)$ is non-empty, compact in H_w , invariant by the flow and it attracts all trajectories in H_w , uniformly for all the initial data \mathbf{u}_0 in a bounded set of H .*

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A Appendix

The following discrete Gronwall lemma can be found in e.g. [Emmrich, 1999, Lemma 3.2], [Tone and Wirosoetisno, 2007]:

Lemma A.1. *Let $\{a_n\}, \{g_n\} \subset \mathbb{R}, \{\tau_n\} \subset \mathbb{R}^+$ be such that*

$$\frac{a_n - a_{n-1}}{\tau_n} \leq g_n + (1 - \theta)\lambda_{n-1}a_{n-1} + \theta\lambda_n a_n, \quad n = 1, 2, \dots$$

If $(1 - \theta\tau_n\lambda_n) > 0$, $1 + (1 - \theta)\lambda_{n-1}\tau_n > 0$, $n = 1, 2, \dots$ then

$$a_n \leq a_0 \prod_{\ell=1}^n \omega_\ell + \sum_{j=0}^{n-1} \frac{\tau_{j+1}g_{j+1}}{1 + (1 - \theta)\lambda_j\tau_{j+1}} \prod_{\ell=j+1}^n \omega_\ell, \quad (A.1)$$

$$\text{where } \omega_\ell = \frac{1 + (1 - \theta)\lambda_\ell\tau_{\ell+1}}{1 - \theta\lambda_{\ell+1}\tau_{\ell+1}}.$$

The following Aubin-Lions compactness result appears in e.g. [Lions, 1969] or [Temam, 2001] :

Lemma A.2. *Let X_0, X and X_1 be three Banach spaces such that $X_0 \subseteq X \subseteq X_1$ and X_i , $i = 1, 2$ are reflexive. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1 . For $p, q > 1$, let*

$$\mathcal{X} = \{u \in L^p(0, T; X_0) \text{ such that } u \in L^q(0, T; X_1)\}.$$

Then the embedding of \mathcal{X} into $L^p(0, T; X)$ is compact.

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